The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #1 ***Solution 1.0***

Date: October 2, 2025 Course: EE 313 Evans

Name: ____
Last, First

- **Exam duration**. The exam is scheduled to last 75 minutes.
- **Materials allowed**. You may use books, notes, your laptop/tablet, and a calculator.
- **Disable all networks**. Please disable all network connections on all computer systems. You may not access the Internet or other networks during the exam.
- **No AI tools allowed**. As mentioned on the course syllabus, you may not use GPT or other AI tools during the exam.
- **Electronics.** Power down phones. No headphones. Mute your computer systems.
- **Fully justify your answers**. When justifying your answers, reference your source and page number as well as quote the content in the source for your justification. You could reference homework solutions, test solutions, etc.
- **Matlab**. No question on the test requires you to write or interpret Matlab code. If you base an answer on Matlab code, then please provide the code as part of the justification.
- **Put all work on the test**. All work should be performed on the quiz itself. If more space is needed, then use the backs of the pages.
- **Academic integrity.** By submitting this exam, you affirm that you have not received help directly or indirectly on this test from another human except the proctor for the test, and that you did not provide help, directly or indirectly, to another student taking this exam.

Problem	*Point Value*	*Your score*	*Topic*
1	24		Sinusoidal Signals
2	26		Fourier Series
3	26		Sampling and Aliasing
4	24		Time-Frequency Analysis
Total	100		

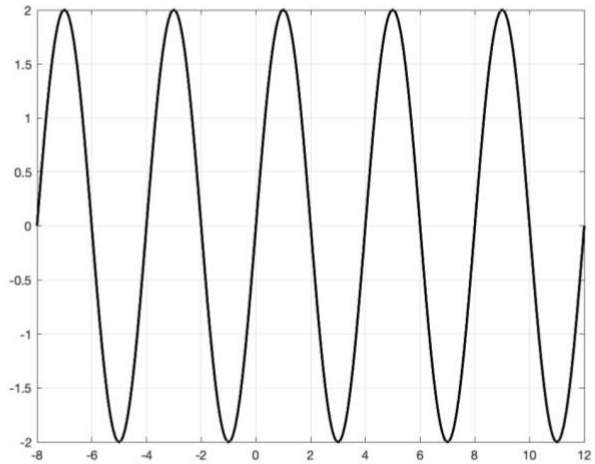
Fall 2024 Midterm 1.1 | SPFirst Sec. 2-1 to 2-3 | Lecture Slide 1-16 | Homework Prob. 1.1

Problem 1.1 *Sinusoidal Signals*. 24 points.

Consider the sinusoidal signal
$x(t) = A \sin(2\,\pi\, f_0\, t\, +\, \theta)$ with

- amplitude A
- continuous-time frequency f_0 in Hz
- phase θ in radians

From the plot of $x(t)$ on the right,



milliseconds

Note: 1 millisecond = 10^{-3} seconds

(a) Estimate the amplitude A. Explain how you estimated the value of this parameter. *6 points.*

In a sinusoidal signal, the peak occurs at A and valley occurs at $-A$ because the values of cosine are in the interval $[-1, 1]$. From the plot, $A = 2$.

(b) Estimate the continuous-time frequency f_0 in Hz. Explain how you estimated the value of this parameter. *6 points.*

***Approach #1*: The duration from one peak to the next is about 4ms and this represents the period T_0 of the sinusoidal signal. The fundamental frequency $f_0 = \frac{1}{T_0} = \frac{1}{0.004s} = 250$ Hz .**

***Approach #2*: From counting the peaks, the plot contains exactly five periods and lasts for $0.020s$. Hence, the fundamental period is $T_0 = 0.004s$ and the fundamental frequency is $f_0 = \frac{1}{T_0} = \frac{1}{0.004s} = 250$ Hz.**

***Approach #3*: The plot contains 10 pairs of zero crossings over the $0.020s$ duration of the signal. Hence, there are 5 periods in $0.020s$. The period is $T_0 = \frac{0.020s}{5} = 0.004s$ and the frequency is $f_0 = \frac{1}{T_0} = \frac{5}{0.020s} = 250$ Hz .**

(c) Estimate the phase θ in radians. Explain how you estimated the value of this parameter. *6 points*.

The amplitude of the sine wave is 0 at $t = 0$, which means $\theta = 0$ rad.

(d) What is the phase of the signal $x(t - 0.001)$? Please show your intermediate steps. *6 points*.

$$x(t - 0.001) = A \sin(2\,\pi\, f_0\, (t - 0.001s)\, +\, \theta) = A \cos(2\,\pi\, f_0\, t - 2\,\pi\, f_0\, (0.001) + \theta)$$

Hence, the phase is $-2\,\pi\, f_0\, (0.001s) + \theta = -2\,\pi\,(250\,Hz)(0.001s) + 0 = -0.5\,\pi$.

SPFirst Sec. 2-3.2 | Lecture Slide 2-4

Problem 1.2. *Fourier Series Properties.* 26 points.

The continuous-time Fourier series has several properties.

In this problem, $x(t)$ is periodic with fundamental frequency f_0 and Fourier series coefficients a_k.

For example, if $y(t) = A\, x(t)$, the Fourier series coefficients b_k for $y(t)$ can be found using $b_k = A\, a_k$:

$$y(t) = A\, x(t) = A \sum_{k=-\infty}^{\infty} a_k\, e^{j2\pi(k f_0)t} = \sum_{k=-\infty}^{\infty} A\, a_k\, e^{j2\pi(k f_0)t}$$

For the following expressions, derive the relationship between the Fourier series coefficients b_k for $y(t)$ and the Fourier series coefficients a_k for $x(t)$ where

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

(a) $y(t) = x_1(t) + x_2(t)$ where $x_1(t)$ has a fundamental frequency f_0 and Fourier series coefficients c_k and $x_2(t)$ has a fundamental frequency f_0 and Fourier series coefficients d_k. *8 points.*

(b) $y(t) = e^{-j\,2\,\pi\, f_0\, t}\, x(t)$. This is a type of amplitude modulation. *9 points.*

(c) $y(t) = x\left(\frac{t}{2}\right)$. *9 points.*

Problem 1.3. *Sampling and Aliasing.* 26 points.

A frequency of 46 kHz is higher than the normal audible range of 20 Hz to 20 kHz for a human being.

Consider a continuous-time signal $x(t) = \cos(2\ \pi\ f_0\ t)$ where $f_0 = 46$ kHz.

Sample the signal using a sampling rate of $f_s = 48$ kHz.

(a) Derive a formula for the discrete-time signal $x[n]$ that results from sampling $x(t)$. *6 points*.

Sampling in the time domain can be modeled as an instantaneous closing and opening of a switch. Each time that the switch is closed, the input is gated to the output. In practice, this could be implemented by a pass transistor with a sampling clock feeding the gate terminal.

$$x[n] = x(t)|_{t=nT_s} = \cos(2\ \pi\ f_0\ (n\ T_s)) = \cos\left(2\ \pi\ f_0\ \left(\frac{n}{f_s}\right)\right) = \cos\left(2\pi\left(\frac{f_0}{f_s}\right)n\right)$$

The discrete-time frequency corresponding to continuous-time frequency f_0 is $\omega_0 = 2\pi\frac{f_0}{f_s}$

(b) Determine the discrete-time frequency to which the continuous-time frequency of f_0 will alias. *9 points*.

Approach #1: Using a time-domain approach.

$$x[n] = \cos\left(2\pi\left(\frac{f_0}{f_s}\right)n\right) = \cos\left(2\pi\left(\frac{46\text{ kHz}}{48\text{ kHz}}\right)n\right) = \cos\left(2\pi\left(\frac{23}{24}\right)n\right)$$

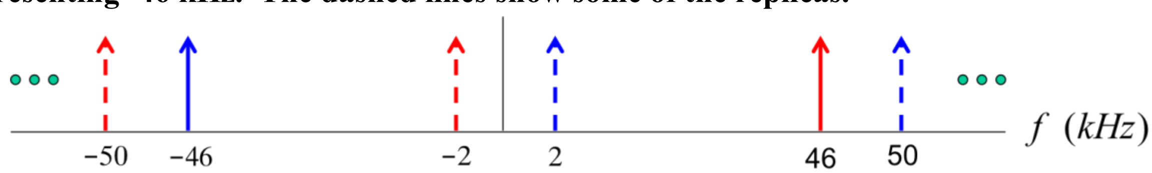
We can subtract an offset in the argument of $2\ \pi\ n$ without changing $x[n]$:

$$\cos\left(2\pi\left(\frac{23}{24}\right)n - 2\pi n\right) = \cos\left(2\pi\left(\frac{23}{24} - 1\right)n\right) = \cos\left(2\pi\left(-\frac{1}{24}\right)n\right) = \cos\left(2\pi\left(\frac{1}{24}\right)n\right)$$

Continuous-time frequency of f_0 will alias to a discrete-time frequency of $2\pi\frac{1}{24}$ rad/sample.

Approach #2*: *Using a frequency-domain approach.

In the frequency domain, sampling of $x(t)$ will include the discrete-time frequencies corresponding to continuous-time frequencies 46 kHz and -46 kHz as well as replicas located at offsets of integer multiples of 2 π rad/sample in discrete-time frequency (where 2π rad/sample corresponds to the sampling rate of 48 kHz in continuous-time frequency). Due to the Sampling Theorem, the reconstructed frequencies are from -½ f_s to ½ f_s and hence the aliased continuous-time frequency is 2 kHz. The continuous-time frequencies of $x(t)$ are shown in solid lines with the red solid line representing 46 kHz and the blue solid line representing -46 kHz. The dashed lines show some of the replicas.



(c) What is the equivalent continuous-time frequency for the aliased discrete-time frequency in (b)? *9 points*.

With $\omega_1 = 2\pi\frac{f_1}{f_s}$ and $f_s = 48$ kHz, we have $f_1 = 2$ kHz.

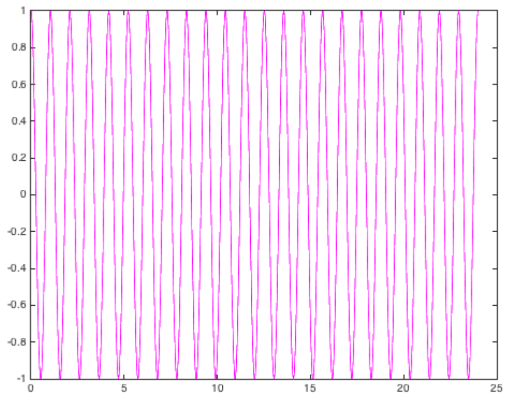
(d) Is the aliased frequency audible? *2 points*.

Yes, the aliased frequency of 2 kHz is in the audible range of 20 Hz to 20 kHz.

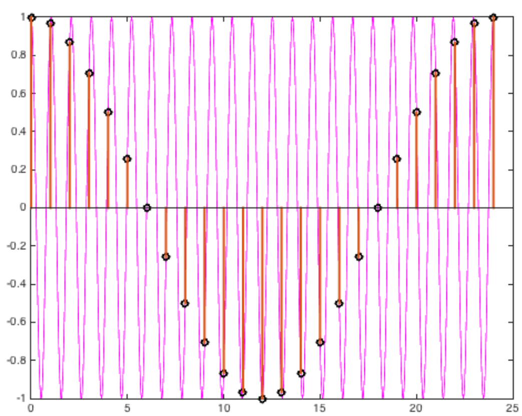
Problem 1.2 Supplemental information not expected for students to have provided in their answers.

Matlab code to show aliasing in time domain

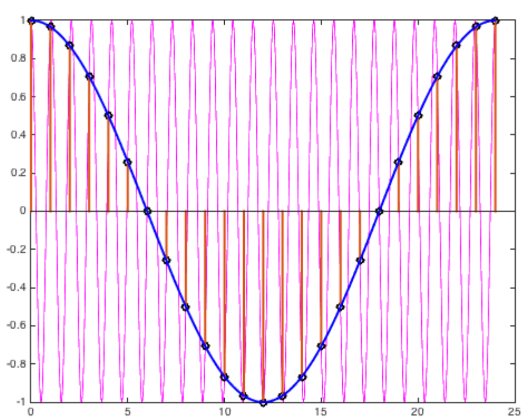
Plot $x(t) = \cos(2\pi f_0 t)$



Plot samples $x(n T_s)$ superimposed on $x(t) = \cos(2\pi f_0 t)$



Plot $x_1(t) = \cos(2\pi f_1 t)$ and $x(n T_s)$ superimposed on $x(t) = \cos(2\pi f_0 t)$



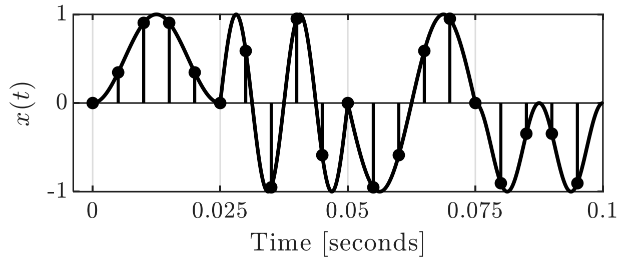
```
%% Part 1: Define Signals
wHat = 2*pi*(1/24);
nmax = 24;
n = 0:nmax;
x1 = cos(wHat*n);
x = cos(2*pi*(23/24)*n);

fs = 1;      %% fs=1 to align DT and CT
f1 = 2/48;   %% Actual fs goes in denom
w1Hat = 2*pi*f1/fs;
period = round(fs/f1);
f0 = 46/48; %% Actual fs goes in denom
w0Hat = 2*pi*f0/fs;
Ts = 1/fs;
tmax = (nmax/period)*(1/f1);
t = 0 : (Ts/100) : tmax;
x1cont = cos(2*pi*f1*t);
xcont = cos(2*pi*f0*t);

%% Part 2: Generate Plots
figure;
plot(t, xcont, 'm-', 'LineWidth', 1);

figure;
plot(t, xcont, 'm-', 'LineWidth', 1);
hold;
stem(n, x1, 'Linewidth', 2,
'MarkerEdgeColor', 'black');
stem(n, x, 'Linewidth', 2,
'MarkerEdgeColor', 'black');

figure;
plot(t, xcont, 'm-', 'LineWidth', 1);
hold;
stem(n, x1, 'Linewidth', 2,
'MarkerEdgeColor', 'black');
stem(n, x, 'Linewidth', 2,
'MarkerEdgeColor', 'black');
plot(t, x1cont, 'b-', 'LineWidth', 2);
```

Problem 1.4. *Time-Frequency Analysis*. 24 points.

This problem is related to mini-project #1. Please justify your answers.

The continuous-time signal $x(t)$ is defined between $0 \le t \le 0.1$. The discrete-time signal $x[n]$ is obtained by sampling $x(t)$ at a rate $f_s = 200$ Hz. The plot of $x(t)$ and $x[n]$ are provided below.

$$x(t) = \begin{cases} \cos^2(2\pi\, 20\, t) & 0 \le t < 0.025 \\ \sin(2\pi\, 80\, t) & 0.025 \le t < 0.05 \\ -\sin(2\pi\, 40\, t) & 0.05 \le t < 0.075 \\ -\cos^2(2\pi\, 40\, t) & 0.075 \le t \le 0.1 \end{cases}$$



A complex image signal $S[m,k]$ is obtained by taking the short-time Fourier transform of $x[n]$ using non-overlapping rectangular windows $w[n]$ of length $N = 5$ samples:

$$\mathbf{S}[m,k] = \mathbf{STFT}\{x[n]\}[m,k] = \sum_{n=0}^{N-1} x[n]\, w[n-m]\, e^{-j2\pi\frac{k}{N}n}$$

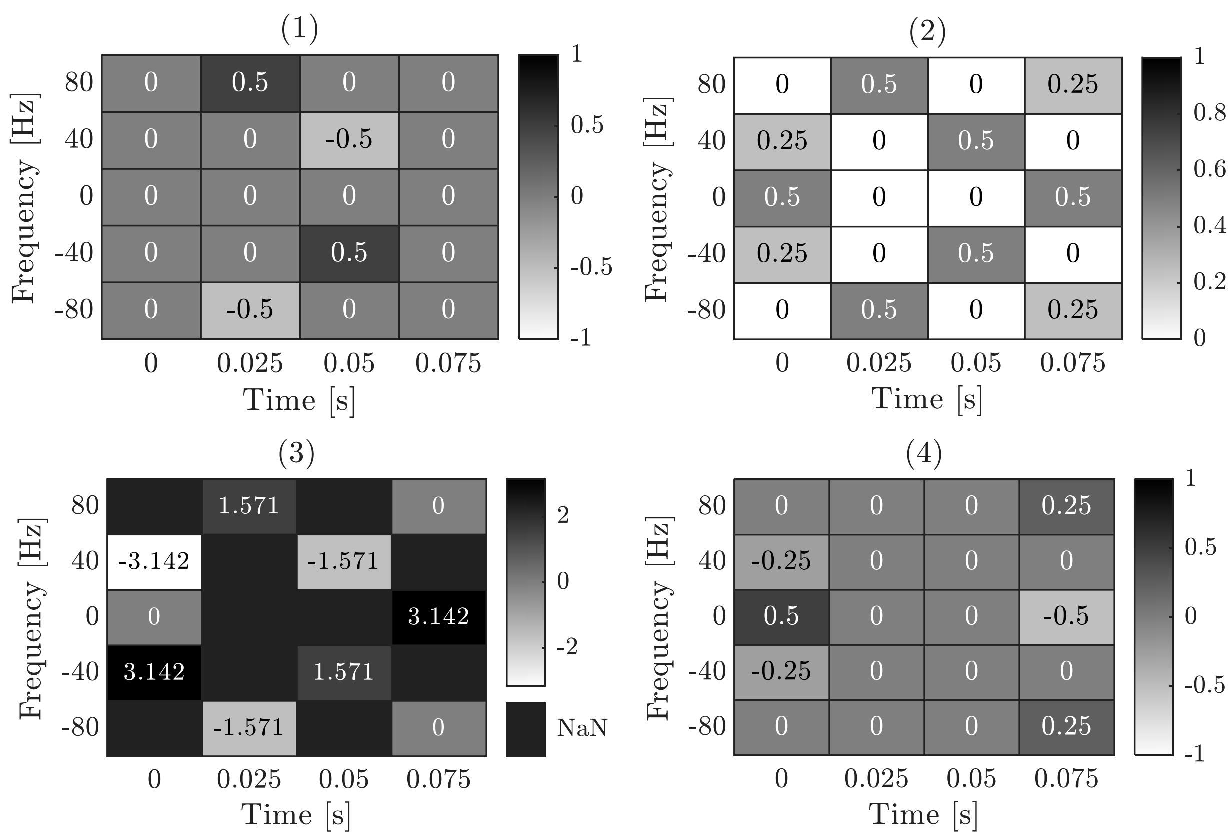
The plots below visualize the components of $\mathbf{S}[m,k]$. Label each plot as one of the four options:

A. Magnitude: $|\mathbf{S}[m,k]|$ **Plot (2)**

B. Phase: $\angle\mathbf{S}[m,k]$ **Plot (3)**

C. Real part: $\mathrm{Re}\{\mathbf{S}[m,k]\}$ **Plot (4)**

D. Imaginary part: $\mathrm{Im}\{\mathbf{S}[m,k]\}$ **Plot (1)**



Justification for the answers are on the next page.

Note: The printed version of the exam contained the following typos:

$\cos^2(2\pi\ 20\ t)$ instead of $\sin^2(2\pi\ 20\ t)$
$-\cos^2(2\pi\ 40\ t)$ instead of $-\sin^2(2\pi\ 40\ t)$

If the STFT images were generated using $x(t)$ as printed originally, the structure of all four STFT components would remain identical, but the plots of the phase, real part, and imaginary part would have slightly different values.

Solution:
The time period from $t = 0$ to $t = 0.1$ corresponds to four windows, each with 5 samples each. $x(t)$ contains different sinusoidal components during each window.

Let $w_1(t) = \mathbf{1}_{[0,0.025)}$, $w_2(t) = \mathbf{1}_{[0.025,0.05)}$, $w_3(t) = \mathbf{1}_{[0.05,0.075)}$, and $w_4(t) = \mathbf{1}_{[0.075,0.1)}$ where

$$\mathbf{1}_{[a,b)} = \begin{cases} 1 & a \le t \le b \\ 0 & \text{otherwise} \end{cases}$$

Then, we can express $x(t)$ as a sum of transient sinusoidal components:

$$x(t) = \underbrace{\sin^2(2\pi\ 20\ t)}_{x_1(t)} w_1(t) + \underbrace{\sin(2\pi\ 80\ t)}_{x_2(t)} w_2(t) + \underbrace{-\sin(2\pi\ 40\ t)}_{x_3(t)} w_3(t) + \underbrace{-\sin^2(2\pi\ 40\ t)}_{x_4(t)} w_4(t)$$

Since $\sin^2\theta = 0.5 - 0.5\cos(2\theta)$,

$$x_1(t) = 0.5 - 0.5\ \cos(2\pi\ 40\ t), \qquad x_4(t) = -0.5 + 0.5\ \cos(2\pi\ 80\ t)$$

By examining the frequency components in each window, we can determine which plots correspond to which STFT components.

Col.	Time [seconds]	Time [samples]	Signal value	Frequencies present
1	$0 \le t < 0.025$	$0 \le n < 5$	$0.5 - 0.5\ \cos(2\pi\ 40\ t)$	0 Hz, $\pm$40 Hz
2	$0.025 \le t < 0.05$	$5 \le n < 10$	$\sin(2\pi\ 80\ t)$	$\pm$80 Hz
3	$0.05 \le t < 0.075$	$10 \le n < 15$	$-\sin(2\pi\ 40\ t)$	$\pm$40 Hz
4	$0.075 \le t < 1$	$15 \le n < 20$	$-0.5 + 0.5\ \cos(2\pi\ 80\ t)$	0 Hz, $\pm$80 Hz

For a real signal, the Fourier series is conjugate symmetric. Therefore, the imaginary component of a real signal is always zero at 0 Hz. Only plot (1) is zero at 0 Hz for all values of time. ***Plot (1) corresponds to the imaginary component.***

The magnitude of a complex number is non-negative. Only plot (2) is non-negative. ***Plot (2) corresponds to the magnitude component.***

The phase is zero for a DC offset and π for a negative DC offset. ***Plot (3) corresponds to the phase.***

The real component at 0 Hz is the average value. The average value of $x(t)$ is 0.5 in the first window, 0 in the second and third windows, and -0.5 in the last window. ***Plot (4) corresponds to the real part.***

Matlab code to produce the plots for problem 1.4.

```
fs = 200;
t = linspace(0,0.1,1000);
nTs = linspace(0,0.1-1/fs,0.1*fs);
w1 = @(t) 1.0*(0<=t & t<0.025);
w2 = @(t) 1.0*(0.025<=t & t<0.05);
w3 = @(t) 1.0*(0.05<=t & t<0.075);
w4 = @(t) 1.0*(0.075<=t & t<=0.1);
x1 = @(t) 0.5-0.5*cos(2*pi*40*t);
x2 = @(t) sin(2*pi*80*t);
x3 = @(t) -sin(2*pi*40*t);
x4 = @(t) -0.5+0.5*cos(2*pi*80*t);
x = @(t) x1(t).*w1(t) + x2(t).*w2(t) + x3(t).*w3(t) + x4(t).*w4(t);
figure; plot(t,x(t),'-k','linewidth',1.2);
hold on; stem(nTs,x(nTs),'filled','k','linewidth',1,'MarkerSize',3);
xlabel('Time [seconds]','Interpreter','latex')
ylabel('$$x(t)$$','Interpreter','latex')
set(gca,'TickLabelInterpreter','latex')
ylim([-1,1])
set(gca,'XTick',[0,0.025,0.05,0.075,0.1]);
grid on;

X = reshape(x(nTs),5,4);
S = (1/5)*fftshift(fft(X),1);
x = [0,0.025,0.05,0.075];
y = [80,40,0,-40,-80];

Sr = real(S); Sr(abs(Sr)<1e-5) = 0;
figure;  heatmap(x,y,Sr);  colormap('gray');  colormap(flipud(colormap));  caxis([-
1,1]);
xlabel('Time [s]'); ylabel('Frequency [Hz]')
set(gca,'Interpreter','latex')
title('(4)')

Si = imag(S); Si(abs(Si)<1e-5) = 0;
figure;  heatmap(x,y,Si);  colormap('gray');  colormap(flipud(colormap));  caxis([-
1,1]);
xlabel('Time [s]'); ylabel('Frequency [Hz]')
set(gca,'Interpreter','latex')
title('(1)')

Sm = abs(S); Sm(abs(S)<1e-5) = 0; Sm(abs(Sm-0.5)<1e-5) = 0.5;
figure;     heatmap(x,y,Sm);     colormap('gray');     colormap(flipud(colormap));
caxis([0,1]);
xlabel('Time [s]'); ylabel('Frequency [Hz]')
set(gca,'Interpreter','latex')
title('(2)')

Sp = angle(S); Sp(abs(Sp)<1e-5) = 0; Sp(abs(S)<1e-5) = nan;
figure;  heatmap(x,y,Sp);  colormap('gray');  colormap(flipud(colormap));  caxis([-
pi,pi]);
xlabel('Time [s]'); ylabel('Frequency [Hz]')
set(gca,'Interpreter','latex')
title('(3)')
```